AN IMPROVED PROCEDURE FOR SELECTING THE BETTER

OF TWO BERNOULLI POPULATIONS*

by

Donald A. Berry and Milton Sobel
University of Minnesota

Technical Report No. 154

December 1971

University of Minnesota
Minneapolis, Minnesota

*This research was supported by National Science Foundation Grant No. 28922X.

1. Introduction.

An inverse sampling procedure R_I which uses the play-the-winner sampling scheme for selecting the better of two Bernoulli populations was considered by Sobel and Weiss [5] and was included in a comparison of several procedures in Hoel and Sobel [3]. A general review of sequential procedures (including R_I) with comparisons is given in [7]. In view of Table 3 in [3] it is apparent that a modification of procedure R_I would be desirable to avoid the property that the expected total number of observations gets large without bound as both probabilities of success approach zero. Procedure R_{IT} studied here is such a modification. Hoel [2] suggests another procedure (R_H) that avoids this property (cf. [2] and Table 3 of [3]).

Bernoulli populations π_1 and π_2 are given with probabilities of success $p_1 \geq p_2$, respectively; we refer to the former as the better and the latter as the poorer population. Let $\Delta = p_1 - p_2$ and let $\bar{p} = \frac{1}{2}(p_1 + p_2)$, so that $p_1 = \bar{p} + \Delta/2$ and $p_2 = \bar{p} - \Delta/2$. The goal is to select one of the two populations and assert that it is the better of the two. For $p_1 > p_2$ we make a correct selection (CS) using procedure R when we select π_1, denoting the probability of CS by $P\{CS\}$, or sometimes by $P\{CS|R\}$. The goal is to find a "good" sampling and terminal selection procedure from the class of procedures R that satisfy the so-called P^*-condition:

$$(1.1) \qquad P\{CS|R\} \geq P^* \quad \text{whenever} \quad \Delta \geq \Delta^*,$$

where $\Delta^* > 0$ and $P^* < 1$ are preassigned constants. Such procedures will be called P^*-admissible. In the class of P^*-admissible procedures a "good" procedure is one that makes some objective (or loss) function small.

Let N_i denote the number of observations on π_i and $N_1 + N_2 = N$. One objective function considered here is the expected total number of observations $E\{N\}$; another is the expected number of observations on the poorer population, $E\{N_2\}$. In [3] Sobel and Weiss compare two inverse sampling sequential procedures, the "play-the-winner" procedure R_I and the "vector-at-a-time" procedure R_I'. Under both procedures sampling continues until one population has r successes and is selected as better. Under R_I', N_1 is always equal to N_2. Under R_I a population is selected at random initially and observed. Subsequently, the same population is used after a success and the other population is used after a failure. It is shown in [5] that R_I is asymptotically ($r \to \infty$) uniformly better than R_I' in both of the above senses: it has a smaller $E\{N\}$ and a smaller $E\{N_2\}$ for all points in the parameter space.

We investigate herein a procedure R_{IT} which modifies R_I by truncating sampling whenever the play-the-winner scheme has gone through c cycles (i.e., whenever c failures have obtained on both populations). Under R_{IT} therefore, sampling terminates whenever either population yields r successes or both yield c failures. In either case the population with the larger number of successes is selected; if the numbers of successes are the same (which can occur only after c cycles) a population is selected randomly. Many pairs (r, c) used in procedure R_{IT} make it P^*-admissible; one obvious such pair has the r used in R_I (call it r_0) and $c = \infty$. One result of this paper is that asymptotically $r = c = r_0$ makes R_{IT} P^*-admissible and uniformly better than R_I in both senses mentioned above, and that this pair is uniformly better than any other P^*-admissible pair. In the exact (non-asymptotic) problem a slightly larger common value of r and c

than r_0 is required under R_{IT} to make it P^*-admissible, thus making $E\{N\}$ and $E\{N_2\}$ slightly larger for some values of $\bar{p}$ under procedure R_{IT}.

Our interest in making $E\{N\}$ small is motivated primarily by problems in which the terminal decision is paramount and not the number of times that a particular population is used during the course of experimentation or the number of failures obtained. In problems such as clinical trials for which both a terminal decision must be made and the poorer population used as infrequently as possible, the experimenter would be primarily interested in making $E\{N_2\}$ small (see also [4]). With such applications in mind we derive exact expressions for and present tables of both $E\{N\}$ and $E\{N_2\}$ under procedure R_{IT}. Problems for which the expected total number of failures during experimentation is to be minimized are "two-armed bandit" problems (see [1]).

2. The Exact $P\{CS\}$ for Procedure R_{IT}.

The probability of a correct selection $P\{CS\}$ can be written as a sum $P + P'$ where P is the probability of selecting π_1 before c cycles (i.e., before c failures on each population under the play-the-winner scheme) and P' is the probability of selecting π_1 based on the number of successes in exactly c cycles. For $i = 1,2$, let $F_i(r)$ denote the number of failures on π_i before the r^{th} success on π_i and let $S_i(c)$ denote the number of successes on π_i before the c^{th} failure on π_i. When there is a tie, i.e., when $S_1(c) = S_2(c) < r$, we select a population at random (by tossing a fair coin) giving 1/2 as the conditional probability of correct selection. When π_1 and π_2 both yield the r^{th} success in the same cycle, i.e., $F_1(r) = F_2(r) < c$, the first one sampled will be selected and the $P\{CS\}$ is again 1/2. Hence,

$$(2.1)\quad P\{CS|R_{IT}\} = P + P' = P\{F_1(r) < F_2(r),\ F_1(r) < c\} + \tfrac{1}{2}P\{F_2(r) = F_1(r) < c\} + P\{S_2(c) < S_1(c) < r\} + \tfrac{1}{2}P\{S_2(c) = S_1(c) < r\}.$$

We shall make use of a basic identity (cf., e.g., equation (2.15) of [5] and (2.9) of [6])

$$p^t \sum_{j=s}^{\infty} \binom{j+t-1}{j} q^j = I_q(s, t) = 1 - p^t \sum_{j=0}^{s-1} \binom{j+t-1}{j} q^j, \tag{2.2}$$

where $0 < p < 1$ and $p + q = 1$.

Using (2.2) and the fact $F_1(r)$, $F_2(r)$, $S_1(c)$, $S_2(c)$ are all negative binomial chance variables, we obtain for P, after letting $F_1(r) = i$,

$$\begin{aligned} P &= p_1^r \sum_{i=0}^{c-1} \binom{i+r-1}{i} q_1^i \left[I_{q_2}(i+1, r) + \left\{ \frac{I_{q_2}(i,r) - I_{q_2}(i+1,r)}{2} \right\} \right. \\ &= p_1^r \sum_{i=0}^{c-1} \binom{i+r-1}{i} q_1^i \left\{ \frac{I_{q_2}(i,r) + I_{q_2}(i+1,r)}{2} \right\}. \end{aligned} \tag{2.3}$$

and for P', after letting $S_1(c) = j$,

$$P' = q_1^c \sum_{j=0}^{r-1} \binom{j+c-1}{j} p_1^j \left\{ \frac{I_{q_2}(c,j) + I_{q_2}(c,j+1)}{2} \right\}. \tag{2.4}$$

Therefore the required $P\{CS\}$ under procedure R_{IT} is given by

$$\begin{aligned} P\{CS|R_{IT}\} &= p_1^r \sum_{i=0}^{c-1} \binom{i+r-1}{i} q_1^i \left\{ \frac{I_{q_2}(i,r) + I_{q_2}(i+1, r)}{2} \right\} \\ &\quad + q_1^c \sum_{j=0}^{r-1} \binom{j+c-1}{j} p_1^j \left\{ \frac{I_{q_2}(c,j) + I_{q_2}(c,j+1)}{2} \right\}. \end{aligned} \tag{2.5}$$

To show the symmetry possessed by (2.5), and for reasons to be pointed out later, we use (2.2) to rewrite P as

$$\begin{aligned} P &= P\{F_1(r) < c \le F_2(r)\} + P\{F_1(r) < F_2(r) < c\} + \tfrac{1}{2}P\{F_1(r) = F_2(r) < c\} \\ &= I_{p_1}(r, c) I_{q_2}(c, r) + p_2^r \sum_{\alpha=0}^{c-1} \binom{\alpha+r-1}{\alpha} q_2^\alpha \left\{ \frac{I_{p_1}(r,\alpha) + I_{p_1}(r,\alpha+1)}{2} \right\}, \end{aligned} \tag{2.6}$$

where the derivation of the last sum is similar to the one in (2.3).

Hence we have for the $P\{CS\}$ under procedure R_{IT}

$$P\{CS|R_{IT}\} = I_{p_1}(r, c)\, I_{q_2}(c, r) + E'_{r,c}\left\{\frac{I_{p_1}(r,X) + I_{p_1}(r,X+1)}{2}\right\} + E_{c,r}\left\{\frac{I_{q_2}(c,Y) + I_{q_2}(c,Y+1)}{2}\right\}, \tag{2.7}$$

where X and Y are negative binomial chance variables with parameters (p_2, r) and (q_1, c), respectively, $E'_{r,c}$ is the sum in (2.6) and $E_{c,r}$ is the sum in (2.4). If $r = c$ the transformation $p_1 \leftrightarrow q_2$ (and $q_1 \leftrightarrow p_2$) leaves the first term of (2.7) invariant and interchanges the last two terms. Thus, for $r = c$, $P\{CS|R_{IT}\}$ is invariant under this transformation and hence symmetrical about $\bar{p} = \frac{1}{2}$ since this transformation takes $\bar{p} = (p_1 + p_2)/2$ into $1 - \bar{p}$. This result and the form (2.7) are both used in the asymptotic analysis of Section 4 to show that the $P\{CS\}$ is minimized (i.e., the so-called "least favorable (LF) configuration" occurs) at $\bar{p} = \frac{1}{3}$ and at $\bar{p} = \frac{2}{3}$.

3. Exact $E\{N\}$ for Procedure R_{IT}.

To obtain the exact $E\{N_2\}$ under procedure R_{IT} we use the method of generating functions. Let $U_f(s, t)$ (resp., $V_f(s, t)$) denote $E\{N_2\}$ if π_1 (resp., π_2) is to be observed next where s more successes on π_1 or t more successes on π_2 or a total of f more failures are needed for termination of sampling (whichever comes first). Hence

$$\tfrac{1}{2}\{U_{2c}(r, r) + V_{2c}(r, r)\} = E\{N_2\}. \tag{3.1}$$

The recursion formulas determining $E\{N_2\}$ are, for $f > 0$, $s > 0$, and $t > 0$,

$$U_f(s, t) = p_1 U_f(s-1, t) + q_1 V_{f-1}(s, t) \tag{3.2}$$

$$V_f(s, t) = p_2 V_f(s, t-1) + q_2 U_{f-1}(s, t) + 1\,. \tag{3.3}$$

The boundary conditions are

$$U_f(0, t) = V_f(s, 0) = U_0(s, t) = V_0(s, t) = 0. \tag{3.4}$$

Let $U = \sum_{f=1}^{\infty} \sum_{s=1}^{\infty} \sum_{t=1}^{\infty} U_f(s, t)x^s y^t z^f$ and similarly for V. From (3.2), (3.3), and (3.4) we have

$$(3.5) \qquad U = p_1 xU + q_1 zV \text{ (i.e., } U = (\frac{qz}{1-xp})V),$$

$$(3.6) \qquad V = p_2 yV + q_2 zU + \frac{xyz}{(1-x)(1-y)(1-z)},$$

which lead to the expression

$$(3.7) \qquad V = \frac{(\frac{x}{1-x})(\frac{y}{1-y})(\frac{z}{1-z})}{(1-p_2 y)[1 - \frac{q_1 q_2 z^2}{(1-p_1 x)(1-p_2 y)}]} = (\frac{x}{1-x})(\frac{y}{1-y})(\frac{z}{1-z}) \sum_{j=0}^{\infty} \frac{(q_1 q_2)^j z^{2j}}{(1-p_1 x)^j (1-p_2 y)^{j+1}}$$

The coefficient of $x^r y^r z^{2c}$, is given by

$$(3.8) \qquad V_{2c}(r, r) = \sum_{j=0}^{[c-1/2]} (q_1 q_2)^j \sum_{\alpha=0}^{r-1} \binom{j+\alpha-1}{\alpha} p_1^{\alpha} \sum_{\beta=0}^{r-1} \binom{j+\alpha}{\alpha} (p_2)^{\beta}$$

$$= \frac{1}{q_2} \sum_{j=0}^{c-1} I_{q_1}(j, r) I_{q_2}(j+1, r),$$

where the last equality follows from (2.2). Similarly, using (3.5) we have

$$(3.9) \qquad U_{2c}(r, r) = \frac{1}{q_2} \sum_{j=0}^{c-1} I_{q_1}(j+1, r) I_{q_2}(j+1, r).$$

Hence, using (3.1), we obtain the desired result

$$(3.10) \qquad E\{N_2\} = \frac{1}{q_2} \sum_{j=0}^{c-1} I_{q_2}(j+1, r) \left\{ \frac{I_{q_1}(j,r) + I_{q_1}(j+1,r)}{2} \right\}.$$

To obtain $E\{N_1\}$, the expected number of observations from π_1, we can follow a similar argument, or we can simply interchange p_1 and q_1 with p_2 and q_2, respectively, giving

$$(3.11) \qquad E\{N_1\} = \frac{1}{q_1} \sum_{j=0}^{c-1} I_{q_1}(j+1, r) \left\{ \frac{I_{q_2}(j,r) + I_{q_2}(j+1,r)}{2} \right\}.$$

The total expected number of observations is

(3.12) $E\{N\} = E\{N_1\} + E\{N_2\}$;

for $p_1 = p_2$ we have $E\{N\} = 2E\{N_2\}$.

4. Asymptotic Analysis.

For large P^* both the r (number of successes) and the c(number of cycles) needed for termination under a P^*-admissible R_{IT} will be large and the $P\{CS|R_{IT}\}$ can be approximated using the standard normal distribution. This approximation gives asymptotically the least favorable configuration and the best relationship between r and c. It will be seen that asymptotically procedure R_{IT} with $r = c = r_0$ is P^*-admissible and is uniformly better (with regard to both $E\{N\}$ and $E\{N_2\}$) than any other R_{IT} procedure that is P^*-admissible. As a trivial consequence $r = c = r_0$ provides an asymptotically minimax solution in the sense that among such procedures it minimizes the maximum value of $E\{N\}$, as well as of $E\{N_2\}$, over all (p_1, p_2) configurations. This minimax property is conjectured to hold under the exact structure of Sections 2 and 3 but has not been shown to hold. That R_{IT} is not uniformly better than R_I in the exact structure can be seen by example; however, when (p_1, p_2) are such that R_I is better it is only slightly better.

The asymptotic structures considered here have $c \to \infty$ for fixed r and $r \to \infty$ for fixed c; we also must investigate $r = c$ as the common value $\to \infty$.

We require the first two moments of the negative binomials $F_1(r)$, $F_2(r)$, $S_1(c)$, and $S_2(c)$:

$$(4.1) \qquad E\{F_i(r)\} = \frac{rq_i}{p_i} \; ; \; E\{S_i(c)\} = \frac{cp_i}{q_i} \; ; \; \sigma^2\{F_i(r)\} = \frac{rq_i}{p_i^2} \; ; \; \sigma^2\{S_i(c)\} = \frac{cp_i}{q_i^2} \; .$$

For standardized random variables, according to the definition of P,

$$(4.2) \qquad P = P\left\{ \frac{F_2(r) - rq_2/p_2}{\sqrt{rq_2}\;/p_2} > \frac{p_2}{p_1}\sqrt{\frac{q_1}{p_1}}\;\; \frac{F_1(r) - rq_1/p_1}{\sqrt{rq_1}\;/p_1} + \frac{r(q_1/p_1 - q_2/p_2)}{\sqrt{rq_2}\;/p_2} \right.$$

$$\left. \frac{F_1(r) - rq_1/p_1}{\sqrt{rq_1}\;/p_1} < \frac{c - rq_1/p_1}{\sqrt{rq_1}\;/p_1} \right\}$$

$$\sim \int_{-\infty}^{(p_1c - rq_1)/\sqrt{rq_1}} [1 - \Phi(x \frac{p_2}{p_1}\sqrt{\frac{q_1}{q_2}} - \frac{\Delta}{p_1}\sqrt{\frac{r}{q_2}})]\, d\Phi(x),$$

where $\sim$ denotes asymptotic equivalence as $r \to \infty$ and $\Delta\sqrt{r}$ approaches a constant (for fixed $\bar{p}$). Replacing x by $-x$, we get the simpler version

$$(4.3) \qquad P \sim \int_{(rq_1 - p_1c)/\sqrt{rq_1}}^{\infty} \Phi(x \frac{p_2}{p_1}\sqrt{\frac{q_1}{q_2}} + \frac{\Delta}{p_1}\sqrt{\frac{r}{q_2}})\, d\Phi(x) .$$

Similarly for P', we obtain

$$(4.4) \qquad P' \sim \int_{-\infty}^{(rq_1 - p_1c)/\sqrt{p_1c}} \Phi(x \frac{q_2}{q_1}\sqrt{\frac{p_1}{p_2}} + \frac{\Delta}{q_1}\sqrt{\frac{c}{p_2}})\, d\Phi(x),$$

where $c \to \infty$ and $\Delta\sqrt{c}$ approaches a constant, and we approximate the $P\{CS\}$ by the sum of the right sides of (4.3) and (4.4).

Assuming $p_1 > 0$, let $c \to \infty$ holding r fixed. Then $P' \to 0$ and

$$(4.5) \qquad P\{CS\} \sim P \sim \int_{-\infty}^{\infty} \Phi(x \frac{p_2}{p_1}\sqrt{\frac{q_1}{q_2}} + \frac{\Delta}{p_1}\sqrt{\frac{r}{q_2}})\, d\Phi(x) = \Phi\left(\frac{\Delta\sqrt{r}}{\sqrt{p_1^2q_2 + p_2^2q_1}} \right) .$$

To minimize the latter for $\Delta \geq \Delta^*$ we set $\Delta = \Delta^*$. The minimization problem is the same as that of Sobel and Weiss [5] and the resulting least favorable configuration has p_1 and p_2 centered about 2/3 and separated by exactly Δ^*. As in [5], the minimum value in (4.5) is

$$(4.6) \qquad P\{CS | \bar{p} = 2/3,\ \Delta = \Delta^*,\ c \to \infty\} = \Phi(\Delta^*\sqrt{27r/8})$$

and setting this equal to P^* gives the required value of r,

$$(4.7)\quad r = r_0 = \frac{8}{27}\frac{\lambda(P^*)}{(\Delta^*)^2},$$

where $\lambda(P^*)$ is the 100 P^*-percentile of the standard normal distribution.

A similar result follows by assuming $q_1 > 0$ (i.e., $p_1 < 1$) and letting $r \to \infty$ holding c fixed. Then $P \to 0$ and

$$(4.8)\qquad P\{CS\} \sim P' \sim \int_{-\infty}^{\infty} \Phi\left(x\frac{q_2}{p_1}\sqrt{\frac{p_1}{p_2}} + \frac{\Delta}{q_1}\sqrt{\frac{c}{p_2}}\right)d\Phi(x) = \Phi\left(\frac{\Delta\sqrt{c}}{\sqrt{p_2q_1^2 + p_1q_2^2}}\right).$$

To minimize this for $\Delta \geq \Delta^*$ we set $\Delta = \Delta^*$ as before, but the LF configuration is now obtained by centering p_1 and p_2 about 1/3 (with difference equal to Δ^*). The resulting minimum is the same as in (4.6) and (4.7) with r replaced by c.

It follows from these two results that the smallest value of r needed (when p_1 and p_2 are centered about 2/3) is the same as the smallest value of c needed (when p_1 and p_2 are centered about 1/3). Since we want to make both r and c as small as possible (and still satisfy the P^*-condition), it is clear that we have to set them equal and define them by (4.7).

For a complete discussion of the problem of choosing r and c, it is necessary to consider the limit of $P + P'$ as $r \to \infty$ when $r = c$. We need only consider the special case $p_1 = q_1 = 1/2$ since for $p_1 \neq 1/2$ either P or $P' \to 0$ and the resulting $P\{CS\}$ is given by (4.8) or (4.5). For $p_1 = 1/2$ and $p_2 = 1/2 - \Delta^*$ we obtain

$$(4.9)\qquad P\{CS\} \sim \int_0^{\infty} \Phi\left(xp_2\sqrt{\frac{2}{q_2}} + 2\Delta^*\sqrt{\frac{r}{q_2}}\right)d\Phi(x) + \int_{-\infty}^{0} \Phi\left(xq_2\sqrt{\frac{2}{p_2}} + 2\Delta^*\sqrt{\frac{r}{p_2}}\right)d\Phi(x),$$

and we will show that this is greater than the right side of (4.6). From the symmetry of $P\{CS|R_{IT}\}$ about $\bar{p} = \frac{1}{2}$ shown in Section 2, we can get an asymptotically equivalent result by setting $p_2 = q_2 = \frac{1}{2}$ and $p_1 = 1 - q_1 = \frac{1}{2} + \Delta^*$ in (4.3) and (4.4) and adding the results for P and P'. This gives for $r = c \to \infty$ and $\Delta^*\sqrt{r} \to$ constant > 0

$$(4.10) \qquad P + P' \sim P \sim \int_{-\infty}^{\infty} \Phi\left(\frac{x}{p_1}\sqrt{\frac{q_1}{2}} + \frac{\Delta^*\sqrt{2r}}{p_1}\right) d\Phi(x) = \Phi\left(\frac{\Delta^*\sqrt{2r}}{\sqrt{p_1^2 + q_1/2}}\right).$$

To show that the latter is greater than the result in (4.6) we need only note that

$$(4.11) \qquad \Phi\left(\frac{\Delta^*\sqrt{r}}{\sqrt{\frac{1}{2}p_1^2 + q_1(\frac{1}{2})^2}}\right) \geq \operatorname*{Min}_{p_1 - p_2 = \Delta^*} \Phi\left(\frac{\Delta^*\sqrt{r}}{\sqrt{q_2 p_1^2 + q_1 p_2^2}}\right)$$

and that the latter was used in going from (4.5) to (4.6) above.

Utilizing the symmetry proved in Section 2 for $r = c$, it now follows that by taking $r = c$ (satisfying (4.7)) there are two LF-configurations:

(i) p_1 and p_2 centered about 2/3, differing by Δ^*,

(ii) p_1 and p_2 centered about 1/3, differing by Δ^*,

at each of which the value of the $P\{CS\}$ is the preassigned P^*. It is clear that both $E\{N\}$ and $E\{N_2\}$ are strictly increasing in r and in c. Taking r and c individually as small as possible without violating the P^*-condition, (1.1), means taking $r = c = r_0$. Therefore, $r = c = r_0$ makes R_{IT} uniformly better for both $E\{N\}$ and $E\{N_2\}$ considerations than all other P^*-admissible pairs (r, c).

5. Numerical Comparisons.

It should be stressed that the arguments in Section 4 are based on asymptotic analyses. In the exact structure of Section 2 the use of $r = c = r_0$ gives a $P\{CS\}$ of slightly less than P^* in neighborhoods of the least favorable configurations $\bar{p} = 1/3$ and 2/3, so that the common value of r

and c must be increased to slightly more than r_0. This means, for example, that for sufficiently large $\bar{p}$, $E\{N\}$ and $E\{N_2\}$ will be slightly larger for the smallest common value of r and c that makes the pair P^*-admissible than for $r = r_0$ and $c = \infty$ (that is, for procedure R_I). It should be clear that in the exact structure of Section 2 no pair (r, c) is uniformly best in any of the senses discussed here. Still, we conjecture that $r = c$, with the common value as small as possible without violating the P^*-condition, minimizes the maximum of both $E\{N\}$ and $E\{N_2\}$, the maximum being taken over $\bar{p}$ with $p_1 - p_2 = \Delta^*$.

This conjecture is supported by Tables 1, 2, and 3 in which R_{IT} and R_I are compared on the bases of $E\{N\}$ and $E\{N_2\}$. However, it was motivated by the observation that $P\{CS|R_{IT}\}$ is symmetric about $\bar{p} = 1/2$ if $r = c$ and asymmetric otherwise. Therefore, the minimum of $P\{CS\}$ for either $\bar{p} < 1/2$ (for $r > c$) or $\bar{p} > 1/2$ (for $r < c$) is unnecessarily large if $r \neq c$, causing both $E\{N\}$ and $E\{N_2\}$ to be unnecessarily large as well.

Exact values of $E\{N\}$ and $E\{N_2\}$ are presented in Tables 1, 2, and 3 for procedures R_I $(r = r_0,\ c = \infty)$ and R_{IT} (with $r = c$) which have $\min P\{CS\} = P^*$ and $\Delta^* = 0.2$. In Table 1 where $P^* = .90$, for example, $r_0 = 12.17$ (a randomization between 12 and 13) while the common value of r and c required in R_{IT} is 12.47. On the other hand, in Table 3 where $P^* = .99$, $r_0 = 40.00$ and the common value of r and c required in R_{IT} is 40.05, now closer to r_0 as expected in view of the asymptotic equality shown in Section 4.

In Tables 1, 2, and 3 the calculations assume that $\Delta = \Delta^*$ so that $p_1 = \bar{p} + 0.1$ and $p_2 = \bar{p} - 0.1$ except for the column headed $E\{N|\Delta = 0\}$. Numbers in this column represent the total expected number of observations when $p_1 = p_2 = \bar{p}$; half of these are from π_1 and half are from π_2.

TABLE 1

Comparison of R_I ($r_0 = 12.17$) and R_{IT} ($r = c = 12.47$)
When $P^* = .90$ and $\Delta^* = 0.2$

	$E\{N_2 \mid \Delta = \Delta^*\}$		$E\{N \mid \Delta = \Delta^*\}$		$E\{N \mid \Delta = 0\}$	
$\bar{p}$	R_I	R_{IT}	R_I	R_{IT}	R_I	R_{IT}
0.0	--	--	--	--	∞	24.9
0.1	49.2	12.5	110.5	28.1	202.8	27.7
0.2	32.2	13.8	72.7	31.6	100.5	31.2
0.3	23.3	15.1	53.5	35.0	66.3	35.4
0.4	17.8	15.3	41.7	36.3	49.0	38.7
0.5	13.9	13.7	33.7	33.4	38.6	37.5
0.6	10.9	11.1	27.6	28.3	31.5	32.3
0.7	8.3	8.5	22.9	23.5	26.2	26.9
0.8	5.7	5.8	18.4	19.1	21.8	22.5
0.9	2.3	2.3	14.1	14.4	17.8	18.4
1.0	--	--	--	--	12.2	12.5

TABLE 2

Comparison of R_I ($r_0 = 20.04$) and R_{IT} ($r = c = 20.24$)
When $P^* = .95$ and $\Delta^* = 0.2$

	$E\{N_2 \mid \Delta = \Delta^*\}$		$E\{N \mid \Delta = \Delta^*\}$		$E\{N \mid \Delta = 0\}$	
$\bar{P}$	R_I	R_{IT}	R_I	R_{IT}	R_I	R_{IT}
0.0	--	--	--	--	∞	40.5
0.1	80.7	20.2	180.9	45.5	348.4	45.0
0.2	52.5	22.5	119.3	51.4	173.0	50.6
0.3	38.2	24.9	88.3	58.0	114.4	57.8
0.4	29.2	25.7	69.1	61.1	84.8	65.3
0.5	22.9	22.7	56.0	55.7	67.0	64.8
0.6	17.7	18.0	46.0	46.6	54.8	55.3
0.7	13.5	13.6	38.2	38.5	45.8	46.3
0.8	8.9	8.9	30.8	31.1	38.4	38.9
0.9	2.5	2.5	22.4	22.6	31.4	31.8
1.0	--	--	--	--	20.0	20.2

TABLE 3

Comparison of R_I (r_0 = 40.00) and R_{IT} ($r = c = 40.05$)
When $P^* = .99$ and $\Delta^* = 0.2$

	$E\{N_2 \mid \Delta = \Delta^*\}$		$E\{N \mid \Delta = \Delta^*\}$		$E\{N \mid \Delta = 0\}$	
$\bar{p}$	R_I	R_{IT}	R_I	R_{IT}	R_I	R_{IT}
0.0	--	--	--	--	∞	80.1
0.1	160.5	40.0	360.5	90.1	725.5	89.0
0.2	104.3	44.5	237.6	101.7	360.9	100.1
0.3	75.6	49.9	175.6	116.4	239.0	114.4
0.4	57.8	52.4	137.8	125.4	177.9	132.5
0.5	45.2	45.3	111.8	111.8	140.9	137.9
0.6	35.2	35.2	92.3	92.4	115.8	115.9
0.7	26.2	26.2	76.1	76.2	97.3	97.4
0.8	16.4	16.5	60.8	60.9	82.5	82.6
0.9	2.5	2.5	42.5	42.6	68.6	68.7
1.0	--	--	--	--	40.0	40.0

References

[1] Berry, Donald A. (1972). A Bernoulli two-armed bandit. Ann. Math. Statist. 43 871-897.

[2] Hoel, D. (1972). An inverse stopping rule for play-the-winner sampling. JASA 67 148-151.

[3] Hoel, D. and Sobel, M. (1972). New sequential procedures for selecting the best of k binomial populations, with tables and comparisons. Berkeley Symposium (to appear).

[4] Sobel, M. and Weiss, G. H. (1970). Play-the-winner sampling for selecting the better of two binomial populations. Biometrika 57 357-365.

[5] Sobel, M. and Weiss, G. H. (1971). Play-the-winner rule and inverse sampling in selecting the better of two binomial populations. JASA 66 545-551.

[6] Sobel, M. and Weiss, G. H. (1972). Play-the-winner rule and inverse sampling for selecting the best of $k \geq 3$ binomial populations. Ann. Math. Statist. (to appear in late 1972).

[7] Sobel, M. and Weiss, G. H. (1972). Recent results on using the play the winner sampling rule with binomial selection problems. Sixth Berkeley Symposium, Vol. I 717-736.